material is connected with the tensors $\bar{p}_i^a$ and q_i^a the relationship (4.4), where

$$p_{0i}^a = \partial U / \partial x_a^i |_{x^i = x_0^i}$$

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ON THE REALIZATION OF HOLONOMIC CONSTRAINTS*

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The idea of realizing holonomic constraints by means of elastic forces was proposed by Lecornu, Klein and Prandtl /1/ when dealing with the paradox of dry friction discovered by Painleve. The general theorem on the realization of holonomic constraints with the help of elastic forces directed towards the configurational manifold of a constrained system was proposed by Courant and was proved in /2/. The generalization of Courant's theorem was considered in /3-5/ by studying the passage to the limit in the case when the velocity of the system at the initial instant is transverse to the manifold defined by the constraint equations. In /2-5/ the assumption that the system in question in conservative is used to a considerable degree.

The main results of the present paper is the fact that the theorem on the passage to the limit holds without assuming that the generalized forces are potential in character. The elastic forces acting on the "free" system have, in general, no limit when the coefficient of elasticity tends to infinity. However, as is shown below, after suitable regularization these forces tend precisely to the reactions of the system with constraints.

1. Initial equations. Let a natural mechanical system be given in $\mathbf{R}^n = \{r\}$, constrained by n_1 ideal holonomic constraints. Let $E(r^{\cdot}, r)$ be the kinetic energy of the system without constraints and let $F(r^{\cdot}, r)$ be the generalized active force. The equations of motion will have the form

$$(\partial E / \partial r^{\cdot})^{\cdot} - \partial E / \partial r = F + R \tag{1.1}$$

where R is the reaction force of the constraints. The constraints define in $\mathbf{R}^n$ a manifold M of dimensions $n_0 = n - n_1$, over which the system must move. In accordance with the axiom of the ideality of the constraints, the 1-form Rdr vanishes on the vectors tangent to M.

We shall consider the problem of realizing the constraints using the force with potential

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NW, where N is a large positive parameter and the function $W(r)$ takes its minimum value on the manifold of constraints. The equations of motion of a constraint-free system have the form

$$(\partial E/\partial r^{\cdot})^{\cdot} - \partial E/\partial r = F - N\partial W/\partial r \tag{1.2}$$

2. ***Formulation of the result.*** Let $r_\infty(t)$, $0 \leqslant t \leqslant 1$, be the motion of the system with constraints and $R_\infty(t)$ the force of reaction along this motion. We assume that the following conditions hold:

1°. The manifold of constraints is four times continuously differentiable and the functions W and E are, respectively, thrice and twice continuously differentiable in some neighbourhood G of the trajectory of motion r_∞ in the configurational space $\mathbf{R}^n$. In some neighbourhood G' of the trajectory of motion $(r_\infty, r_\infty^{\cdot})$ in the phase space $\mathbf{R}^{2n}$ the function F is twice continuously differentiable.

2°. The function W is non-negative and vanishes on M. The second differential of W is positive definite at every point M of any subspace of dimensions n_1, transverse to the manifold M.

Let the manifold M be defined by the equations $f_k(r) = 0$ $(k = 1, \ldots, n_1)$ and let the differentials df_k be linearly independent at points belonging to M. Then we can take as the function W, the function $N\sum c_k f_k^2$, for example, where c_k are positive constants.

Let $r_N(t)$ be the motion of the constraint-free system with initial conditions $r_N(0) = r_\infty(0)$, $r_N^{\cdot}(0) = r_\infty^{\cdot}(0)$.

Theorem. For sufficiently large N and $0 \leqslant t \leqslant 1$ the motion is well-defined and the following relations hold:

$$r_N(t) = r_\infty(t) + O(N^{-1}),\quad r_N^{\cdot}(t) = r_\infty^{\cdot}(t) + O(N^{-1/2}) \tag{2.1}$$

When $t_{1,2} \in [0, 1]$, we have

$$\int_{t_1}^{t_2} \left(N\frac{\partial W}{\partial r} + R_\infty(t)\right) dt = O(N^{-1/2}) \tag{2.2}$$

along $r_N(t)$.

Note. The second estimate of (2.1) can be refined:

$$r_\parallel^{\cdot}(t) = r_\infty^{\cdot}(t) + O(N^{-1}),\quad r_\perp^{\cdot}(t) = O(N^{-1/2}) \tag{2.3}$$

where $r_\parallel^{\cdot}(t)$ is the orthogonal projection of $r_N^{\cdot}(t)$ on the tangential plane to M at the point $r_\infty(t)$; $r_\perp^{\cdot}(t) = r_N^{\cdot}(t) - r_\parallel^{\cdot}(t)$. The orthogonality is determined using the scalar product, specified by the quadratic form of the energy E.

By (2.2) we have

$$\lim_{t_1, t_2 \to t_0} \lim_{N\to\infty} \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} N\frac{\partial W}{\partial r}(r_N(t))\, dt = -R_\infty(t_0)$$

In the general case, passages to the limit with respect to time and the parameter N are not interchangeable, since, as a rule, there is no limit value for the elastic force $N\partial W/\partial r$ as $N \to \infty$.

In order to illustrate this, we shall consider the motion of a material point of unit mass over the Euclidean plane $\mathbf{R}^2 = \{x, y\}$. Let M be given by the equation $y = 0$, and let the projection of the force F on to the x and y axes be equal to y and 1 respectively. We shall write $W = y^2/2$. Then Eqs.(1.2) will take the form $x^{\cdot\cdot} = y$, $y^{\cdot\cdot} = 1 - Ny$. Since $y_N(0) = y_N^{\cdot}(0) = 0$, we have $y_N(t) = [1 - \cos(N^{1/2}t)]/N$. It is clear that for fixed values of t

$$y_N(t) = O(N^{-1}),\quad y_N^{\cdot}(t) = O(N^{-1/2})$$

but the force $-NW' = -Ny_N = \cos(N^{1/2}t) - 1$ oscillates rapidly (with frequency $N^{1/2}$) about its mean value equal to force of reaction of the constraint $y = 0$. After averaging over time, the oscillations are of the order of $O(N^{-1/2})$ and hence tend to zero as $N \to \infty$. The x coordinate describes the motion along the manifold M. Since $x_N^{\cdot\cdot} = y_N$, we have $x_N - x_\infty = O(N^{-1})$, $x_N^{\cdot} - x_\infty^{\cdot} = O(N^{-1})$. The above example also shows that the estimates (2.1)-(2.3) cannot be improved.

3. ***Proof.*** We can introduce in the neighbourhood of each point of the manifold M new coordinates $x \in \mathbf{R}^{n_0}$, $q \in \mathbf{R}^{n_1}$ in such a manner that M will be specified by the equation $q = 0$ and the quadratic form E will contain the derivatives $x^{\cdot}$ and $q^{\cdot}$ when $q = 0$. To simplify the argument we shall assume that the coordinates are introduced globally in the region G. Then we can assume that r are already such coordinates, while x and q are, respectively, the first n_0 and subsequent n_1 components of r. The equation of motion will not take the form

$$\left(\frac{\partial E}{\partial q^{\cdot}}\right)^{\cdot}-\frac{\partial E}{\partial q}+N\frac{\partial W}{\partial q}=Q,\quad \left(\frac{\partial E}{\partial x^{\cdot}}\right)^{\cdot}-\frac{\partial E}{\partial x}+N\frac{\partial W}{\partial x}=Z \tag{3.1}$$

$$E=T(x^{\cdot},x)+{}^1/_2q^{\cdot}\cdot A(x)\,q^{\cdot}+O(|q|)$$

$$W={}^1/_2q\cdot B(x)\,q+O(|q|^3),\quad Z=X(x^{\cdot},x)+O(|q^{\cdot}|)+O(|q|) \tag{3.2}$$

The matrices A and B are positive definite.

We will introduce the moment

$$p=\partial E/\partial q^{\cdot},\quad y=\partial E/\partial x^{\cdot} \tag{3.3}$$

and write the equations of motion in the form

$$q^{\cdot}=\frac{\partial E}{\partial p},\quad p^{\cdot}=-\frac{\partial E}{\partial q}-N\frac{\partial W}{\partial q}+Q \tag{3.4}$$

$$x^{\cdot}=\frac{\partial E}{\partial y},\quad y^{\cdot}=-\frac{\partial E}{\partial x}-N\frac{\partial W}{\partial x}+Z$$

Here the energy E is assumed to be expressed in terms of q, p, x, y.

The system (3.4) is defined in the region $D'\subset \mathbb{R}^{2n}$, which is an image of the region G' under the mapping $(q, q^{\cdot}, x, x^{\cdot})\mapsto(q, p, x, y)$.

We shall write $\xi=qN^{1/2}$ and assume that $|\xi|<1$. Then from (3.2)-(3.4) we obtain

$$\xi^{\cdot}=N^{1/2}A^{-1}p+f(\xi,p,x,y,N),\quad p^{\cdot}=-N^{1/2}B\xi+g(\xi,p,x,y,N) \tag{3.5}$$

$$x^{\cdot}=O(1),\quad y^{\cdot}=O(1)$$

The functions f, g and their derivatives are of order $O(1)$.

Let $\xi_s(x,y,N)$, $p_s(x,y,N)$ be the solution of a system of equations corresponding to equating the right-hand sides of the firt two equations of system (3.5) to zero. The functions ξ_s, p_s and their derivatives are quantities of the order of $O(N^{-1/2})$.

Let us put $\Xi=\xi-\xi_s$, $P=p-p_s$ and introduce the positive definite quadratic form of the variables Ξ, P:

$$U={}^1/_2P\cdot A^{-1}P+{}^1/_2\Xi\cdot B\Xi$$

Differentiating U we obtain, by virtue of the equations of motion,

$$dU/dt=O(N^{-1/2}U^{1/2})+O(U) \tag{3.6}$$

We will now estimate the motion $r_N(t)$ of the system (1.2) introduced above. We shall choose any $\tau\in[0,1]$ such that for $0\leqslant t\leqslant\tau$ the motion in question is defined and the point $(r, r^{\cdot})$ does not leave G', while the point (q,p,x,y) constructed on it does not leave $D'\cap\{|\xi|<1\}$. Then the quantities Ξ, P, U will be defined for this motion. At the initial instant $(t=0)$ we have $\Xi=O(N^{-1/2})$, $P=O(N^{-1/2})$. From (3.6) we find that along the motion $U=O(N^{-1})$, $\Xi=O(N^{-1/2})$, $P=O(N^{-1/2})$ and therefore

$$\xi=O(N^{-1/2}),\ p=O(N^{-1/2}),\ q=O(N^{-1}),\ q^{\cdot}=O(N^{-1/2})$$

Using these estimates we find from (3.1) and (3.2) that when $0\leqslant t\leqslant\tau$,

$$x^{\cdot}=\partial T/\partial y+O(N^{-1}),\ y^{\cdot}=-\partial T/\partial x+X+h(x,y)\,q^{\cdot}+O(N^{-1}) \tag{3.7}$$

The function h and its derivatives are of order $O(1)$.

Let us introduce $Y=y+h(x,y)\,q$. We obtain

$$x^{\cdot}=\partial T/\partial Y+O(1/N),\ Y^{\cdot}=-\partial T/\partial x+X+O(1/N) \tag{3.8}$$

In the above expression, $x^{\cdot}$ in the arguments of T, X must be expressed in terms of y, x, and y replaced by Y. Neglecting in (3.8) terms or order $O(1/N)$, we arrive at the equation of motion of the system with constraints. The neglected terms are capable of displacing the solution, over the time $\tau\leqslant 1$, only by an amount of the order of $O(1/N)$. The change in initial conditions by a quantity of the order of $O(1/N)$ also shifts the solution during its passage from y to Y by the order of $O(1/N)$. Therefore x, y and hence $x^{\cdot}$ differ, in the motion in question, from the corresponding quantities for the motion r_∞ by $O(1/N)$. Therefore, when $0\leqslant t\leqslant\tau$, the estimates (2.1) and (2.3) of the theorem and subsequent notes also hold. By virtue of these estimates the point $(r_N(t), r_N^{\cdot}(t))$ appears, for $0\leqslant t\leqslant\tau$, at a positive distance from the boundary of G' and $|\xi(t)|<{}^1/_2$ holds. Therefore we can choose $\tau=1$.

Let us now derive the estimate (2.2) of the theorem. Since $\partial W/\partial r$, R_∞ are not invariant under the coordinate change, we shall not use the particular choice of r made above, and we shall assume that $r=r(x,q)$.

Using the estimates (2.1) already proved, we obtain

$$N\frac{\partial W}{\partial r}=N\frac{\partial W}{\partial q}\frac{\partial q}{\partial r}+N\frac{\partial W}{\partial x}\frac{\partial x}{\partial r}=N\frac{\partial W}{\partial q}\frac{\partial q}{\partial r}+O\left(\frac{1}{N}\right)=$$

$$\left[\left(-\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}+\frac{\partial E}{\partial q}+Q\right)\frac{\partial q}{\partial r}\right]+O\left(\frac{1}{N}\right)$$

$$R_\infty=\left[\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}\right]_\infty+\left[\left(\frac{d}{dt}\frac{\partial E}{\partial x^{\cdot}}-\frac{\partial E}{\partial x}-Z\right)\frac{\partial x}{\partial r}\right]_\infty$$

The subscript ∞ means that the corresponding quantity is calculated along the motion of the system with constraints. The last term in the expression for R_∞ is identically equal to zero, since the condition that the expression within the parenthesis in this term vanishes when $q=q^{\cdot}=q^{\cdot\cdot}=0$ is Lagrange's equation for motion with constraints. Using (2.1) again we now obtain

$$N\frac{\partial W}{\partial r}+R_\infty=-\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}+\left[\left(\frac{d}{dt}\frac{\partial E}{\partial q^{\cdot}}-\frac{\partial E}{\partial q}-Q\right)\frac{\partial q}{\partial r}\right]_\infty+$$
$$O\left(\frac{1}{N}\right)=A(x_\infty)\,q^{\cdot\cdot}\cdot\left(\frac{\partial q}{\partial r}\right)_\infty+(N^{-1/2})$$

Integrating from the left and right with respect to t from t_1 to t_2, using integration by parts from the right, and taking into account the fact that $q^{\cdot}=O(N^{-1/2})$, $r_\infty^{\cdot}=O(1)$, we obtain the estimate (2.2) of the theorem.

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THE ASYMPTOTIC STABILIZABILITY OF POSITIONS OF RELATIVE EQUILIBRIUM OF A SATELLITE - GYROSTAT*

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A theorem proved in /1/ is used to study the possibility of asymptotic stabilization of the equilibrium orientations of a satellite-gyrostat using control moments applied to the rotors.

The asymptotic stabilizability of the stationary motions of mechanical systems with cyclic coordinates was also discussed in /2/, where the sufficient condition of stability was formulated. This, as well as the analogous condition of /1/, follows from the classical theory on the sufficient conditions of stabilization /3/. However, in the theorem in /1/ the condition in question leads, by virtue of taking into account the specific features of the systems with cyclic coordinates, to the study of the rank of a matrix of lower dimensions. From this point of view the theorem in /1/ is more suitable for use when studying the stabilizability of the stationary motions of specific mechanical systems.

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